

Credit Risk Models II: Structural Models

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Abstract

This report reviews the structural approach for credit risk modelling, both considering the case of a single firm and the case with default dependences between firms. In the single firm case, we review the Merton (1974) model and first passage models, examining their main characteristics and extensions. Liquidation process models extend first passage models to account for the possibility of a lengthy liquidation process which might or might not end up in default. Finally, we review structural models with state dependent cash flows (recession vs. expansion) or debt coupons (rating-based).

The estimation of structural models is addressed, covering the different ways proposed in the literature.

In the second part of the text, we present some approaches to model default dependences between firms. They account for two types of default correlations: cyclical default correlation and contagion effects. We close the paper with a brief mention of factor models.

The paper pretends to be a guide to the literature, providing a comprehensive list of references and, along the way, suggesting different possible extensions for its future development.

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This paper is part of a series of surveys on credit risk modelling and pricing. The complete list of surveys is available at www.abelelizalde.com, and consists on the following:

1. Credit Risk Models I: Default Correlation in Intensity Models.
2. Credit Risk Models II: Structural Models.
3. Credit Risk Models III: Reconciliation Reduced-Structural Models.
4. Credit Risk Models IV: Understanding and pricing CDOs.
5. Credit Default Swap Valuation: An Application to Spanish Firms.

1 Introduction

There are two primary types of models that attempt to describe default processes in the credit risk literature: structural and reduced form models.

Structural models use the evolution of firms' structural variables, such as asset and debt values, to determine the time of default. Merton's model (1974) was the first modern model of default and is considered the first structural model. In Merton's model, a firm defaults if, at the time of servicing the debt, its assets are below its outstanding debt. A second approach, within the structural framework, was introduced by Black and Cox (1976). In this approach defaults occur as soon as firm's asset value falls below a certain threshold. In contrast to the Merton approach, default can occur at any time.

Reduced form models do not consider the relation between default and firm value in an explicit manner. In contrast to structural models, the time of default in intensity models is not determined via the value of the firm, but it is the first jump of an exogenously given jump process. The parameters governing the default hazard rate are inferred from market data.¹

Structural default models provide a link between the credit quality of a firm and the firm's economic and financial conditions. Thus, defaults are endogenously generated within the model instead of exogenously given as in the reduced approach. Another difference between the two approaches refers to the treatment of recovery rates: whereas reduced models exogenously specify recovery rates, in structural models the value of the firm's assets and liabilities at default will determine recovery rates.

The structural literature on credit risk starts with the paper by Merton (1974), who applies the option pricing theory developed by Black and Scholes (1973) to the modelling of a firm's debt. In Merton's model, the firm's capital structure is assumed to be composed by equity and a zero-coupon bond with maturity T and face value

¹For a review of reduced form models see Elizalde (2005a).

of D . The firm's equity is simply a European call option with maturity T and strike price D on the asset value and, therefore, the firm's debt value is just the asset value minus the equity value. This approach assumes a very simple and unrealistic capital structure and implies that default can only happen at the maturity of the zero-coupon bond.

The paper by Black and Cox (1976) is the first of the so-called *First Passage Models* (FPM). First passage models specify default as the first time the firm's asset value hits a lower barrier, allowing default to take place at any time. When the default barrier is exogenously fixed, as in Black and Cox (1976) and Longstaff and Schwartz (1995), it acts as a safety covenant to protect bondholders. Alternatively it can be endogenously fixed as a result of the stockholders' attempt to choose the default threshold which maximizes the value of the firm (cf. Leland 1994 and Leland and Toft 1996.)

Structural models have considered interest rates both as non-stochastic processes (Black and Cox 1976, Geske 1977, Leland 1994 and Leland and Toft 1996) and as stochastic processes (Ronn and Verma 1986, Kim, Ramaswamy and Sundaresan 1993, Nielsen et al. 1993, Longstaff and Schwartz 1995, Briys and de Varenne 1997 and Hsu, Saá-Requejo and Santa-Clara 2004).²

In First Passage Models, by definition, default occurs the first time the asset value goes below a certain lower threshold, i.e. the firm is liquidated immediately after the default event. In contrast with First Passage Models, a new set of models is being put forward, supported by recent theoretical and empirical research, where a default event does not immediately cause liquidation but it represents the beginning of a

²We reproduce here an updated list of extensions and improvements within the literature of structural models provided by Eom, Helwege and Huang (2003, p. 500): "See, for example, Black and Cox (1976), Bryis and De Varenne (1997), Goldstein, Ju and Leland (2001), Ho and Singer (1982), Kim, Ramaswamy and Sundaresan (1993), Leland (1994, 1998), Nielsen, Saá-Requejo and Santa-Clara (1993), and Titman and Torous (1989). Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997) incorporate strategic defaults into traditional structural models. See also Acharya and Carpenter (2002), Acharya et al. (2000), Anderson, Sundaresan and Tychon (1996), Fan and Sundaresan (2000), and Huang (1997). Duffie and Lando (2001) take into account incomplete accounting information. Garbade (1999) examines managerial discretion. Huang and Huang (2002) and Zhou (2001) incorporate jumps".

process, the liquidation process, which might or might not cause liquidation after it is completed. This practice is consistent, for example, with Chapter 11 of the US Bankruptcy Law, where firms filing for bankruptcy are granted a court-supervised grace period (up to several years) aimed at sorting out their financial problems in order to, if possible, avoid liquidation. We label those models *Liquidation Process Models* (LPM).

State Dependent Models (SDM) represent, together with LPM, two recent efforts to incorporate into structural models different real-life phenomena. Although theoretically they make good sense, they lack empirical research testing its performance. SDM assume that some of the parameters governing the firm's ability to generate cash flows or its funding costs are state dependent, where states can represent the business cycle (recession vs. expansion) or the firm's external rating.

After the single firm case, we review some structural models for default correlations, in order to account for both cyclical default correlation (Zhou 2001b and Giesecke 2004) as well as credit risk contagion effects (Giesecke 2004 and Giesecke and Goldberg 2004). We will finish the default correlation section mentioning the so-called factor models (Schönbucher 2000, Finger 1999, and Frey, McNeil and Nyfeler 2001).

We concentrate on the review of the dynamics of the processes which generate the default times, without paying attention to the valuation formulas for defaultable bonds that each model generates. The aim of this paper is to serve as an introduction and guide to the literature of structural credit risk models. We provide an extensive list of references for each model specification and possible extensions or related papers.

2 Single firm

We denote the physical and risk neutral probability measures as $\tilde{\mathbf{P}}$ and \mathbf{P} respectively, and assume an arbitrage free market.³ Unless otherwise stated, all probabilities and

³For our purposes we shall use the class of equivalent probability measures \mathbf{P} where non-dividend paying asset processes discounted with the risk free interest rate are \mathbf{P} -martingales. See Elizalde

expectations are taken under the risk neutral measure. The model for the default-free term structure of interest rates is given by a short-rate process r_t .

2.1 Merton's model

Merton (1974) makes use of the Black and Scholes (1973) option pricing model to value corporate liabilities. This is an straightforward application only if we adapt the firm's capital structure and the default assumptions to the requirements of the Black-Scholes model. Let us assume that the capital structure of the firm is comprised by equity and by a zero-coupon bond with maturity T and face value of D , whose values at time t are denoted by E_t and $z(t, T)$ respectively, for $0 \leq t \leq T$. The firm's asset value V_t is simply the sum of equity and debt values. Under these assumptions, equity represents a call option on the firm's assets with maturity T and strike price of D . If at maturity T the firm's asset value V_T is enough to pay back the face value of the debt D , the firm does not default and shareholders receive $V_T - D$. Otherwise ($V_T < D$) the firm defaults, bondholders take control of the firm, and shareholders receive nothing. Implicit in this argument is the fact that the firm can only default at time T . This assumption is important to be able to treat the firm's equity as a vanilla European call option, and therefore apply the Black-Scholes pricing formula.

The rest of assumptions Merton (1974) adopts are the inexistence of transaction costs, bankruptcy costs, taxes or problems with indivisibilities of assets; continuous time trading; unrestricted borrowing and lending at a constant interest rate r ; no restrictions on the short selling of the assets; the value of the firm is invariant under changes in its capital structure (Modigliani-Miller Theorem) and that the firm's asset value follows a diffusion process.

The firm's asset value is assumed to follow a diffusion process given by

$$dV_t = rV_t dt + \sigma_V V_t dW_t, \quad (1)$$

where σ_V is the (relative) asset volatility and W_t is a Brownian motion.⁴

(2005a, Appendix A) for details.

⁴Since we are working under the risk neutral probability measure, the drift term of the asset

The payoffs to equityholders and bondholders at time T under the assumptions of this model are respectively, $\max\{V_T - D, 0\}$ and $V_T - E_T$, i.e.

$$E_T = \max\{V_T - D, 0\}, \quad (2)$$

$$z(T, T) = V_T - E_T. \quad (3)$$

Applying the Black-Scholes pricing formula, the value of equity at time t ($0 \leq t \leq T$) is given by

$$E_t(V_t, \sigma_V, T - t) = e^{-r(T-t)} \left[e^{r(T-t)} V_t \Phi(d_1) - D \Phi(d_2) \right] \quad (4)$$

where $\Phi(\cdot)$ is the distribution function of a standard normal random variable and d_1 and d_2 are given by

$$d_1 = \frac{\ln\left(\frac{e^{r(T-t)} V_t}{D}\right) + \frac{1}{2} \sigma_V^2 (T - t)}{\sigma_V \sqrt{T - t}}, \quad (5)$$

$$d_2 = d_1 - \sigma_V \sqrt{T - t}. \quad (6)$$

The probability of default at time T is given by

$$P[V_T < D] = \Phi(-d_2). \quad (7)$$

Therefore, the value of the debt at time t is $z(t, T) = V_t - E_t$.

In order to implement Merton's model we have to estimate the firm's asset value V_t , its volatility σ_V (both unobservable processes), and we have to transform the debt structure of the firm into a zero-coupon bond with maturity T and face value D .

The maturity T of the zero-coupon bond can be chosen either to represent the maturity structure of the debt, for example as the Macaulay duration of all the liabilities, or simply as a required time horizon (for example, in case we are pricing a credit derivative with some specific maturity).

value process is given by the risk-free instantaneous interest rate. Under the physical probability measure, r would be replaced by a parameter μ_V representing the mean rate of return on assets; and the firm's asset process would be given by

$$dV_t = \mu_V V_t dt + \sigma_V V_t d\tilde{W}_t,$$

where \tilde{W}_t is a Brownian motion under the physical probability measure $\tilde{\mathbf{P}}$.

2.1.1 Criticisms and extensions

The main advantage of Merton's model is that it allows to directly apply the theory of European options pricing developed by Black and Scholes (1973). But to do so the model needs to make the necessary assumptions to adapt the dynamics of the firm's asset value process, interest rates, and capital structure to the requirements of the Black-Scholes model. There is a trade off between realistic assumptions and ease of implementation, and Merton's model opts for the latter one. All extensions to this model introduce more realistic assumptions trying to end up with a model not too difficult to implement and with closed, or at least numerically feasible, solutions for the expressions of the debt value and the default probabilities. Merton himself (Merton 1974) presents some extensions to the model, in order to account for coupon bonds, callable bonds, stochastic interest rates, and relaxing the assumption that the Modigliani-Miller Theorem holds.

One problem of Merton's model is the restriction of default time to the maturity of the debt, ruling out the possibility of an early default, no matter what happens with the firm's value before the maturity of the debt. If the firm's value falls down to minimal levels before the maturity of the debt but it is able to recover and meet the debt's payment at maturity, the default would be avoided in Merton's approach.

Another handicap of the model is that the usual capital structure of a firm is much more complicated than a simple zero-coupon bond. Geske (1977, 1979) considers the debt structure of the firm as a coupon bond, in which each coupon payment is viewed as a compound option and a possible cause of default. At each coupon payment, the shareholders have the option either to make the payment to bondholders,⁵ obtaining the right to control the firm until the next coupon, or to not make the payment, in which case the firm defaults. Geske also extends the model to consider characteristics such as sinking funds, safety covenants, debt subordination, and payout restrictions.

The assumption of a constant and flat term structure of interest rates is other

⁵Since shareholders finance each coupon issuing new equity, the dilution effect reduces the relative value of each share.

major criticism the model has received. Jones et al. (1984, p. 624) suggest that “there exists evidence that introducing stochastic interest rates, as well as taxes, would improve the model’s performance.” Stochastic interest rates allow to introduce correlation between the firm’s asset value and the short rate, and have been considered, among others, by Ronn and Verma (1986), Kim, Ramaswamy and Sundaresan (1993), Nielsen et al. (1993), Longstaff and Schwartz (1995), Briys and de Varenne (1997), and Hsu, Saá-Requejo and Santa-Clara (2004).

Another characteristic of Merton’s model, which will also be present in some of the FPM, is the predictability of default. Since the firm’s asset value is modelled as a geometric Brownian motion and default can only happen at the maturity of the debt, it can be predicted with increasing precision as the maturity of the debt comes near. As a result, in this approach default does not come as a surprise, which makes the models generate very low short-term credit spreads.⁶ As we shall review, introducing jumps in the process followed by the asset value has been one of the solutions considered to this problem.

Delianedis and Geske (2001) study the proportion of the credit spread that, in a corporate bond data set, is explained by default risk, using the Merton (1974) and Geske (1977) frameworks. They conclude that it only explains a small fraction of the credit spreads; the rest is attributable to taxes, jumps, liquidity and market risk factors. They also include a jump component in the Merton model finding that (p. 24) “while jumps may explain a portion of the residual spread it is unlikely that jumps can explain it entirely.”

2.2 First passage models

First Passage Models (FPM) were introduced by Black and Cox (1976) extending the Merton model to the case when the firm may default at any time, not only at the maturity date of the debt.

Consider, as in the previous section, that the dynamics of the firm’s asset value

⁶See Jones et al. (1984) and Franks and Torous (1989).

under the risk neutral probability measure \mathbf{P} are given by the diffusion process

$$dV_t = rV_t dt + \sigma_V V_t dW_t, \quad (8)$$

and that there exists a lower level of the asset value such that the firm defaults once it reaches this level. Although Black and Cox (1976) considered a time dependent default threshold, let us assume first a constant default threshold $K > 0$. If we are at time $t \geq 0$ and default has not been triggered yet and $V_t > K$, then the time of default τ is given by

$$\tau = \inf \{s \geq t \mid V_s \leq K\}. \quad (9)$$

Using the properties of the Brownian motion W_t , in particular the reflection principle, we can infer the default probability from time t to time T :⁷

$$P[\tau \leq T \mid \tau > t] = \Phi(h_1) + \exp\left\{2\left(r - \frac{\sigma_V^2}{2}\right) \ln\left(\frac{K}{V_t}\right) \frac{1}{\sigma_V^2}\right\} \Phi(h_2), \quad (10)$$

where

$$h_1 = \frac{\ln\left(\frac{K}{e^{r(T-t)}V_t}\right) + \frac{\sigma_V^2}{2}(T-t)}{\sigma_V\sqrt{T-t}}, \quad (11)$$

$$h_2 = h_1 - \sigma_V\sqrt{T-t}, \quad (12)$$

FPM have been extended to account for stochastic interest rates, bankruptcy costs, taxes, debt subordination, strategic default, time dependent and stochastic default barrier, jumps in the asset value process, etc. Although these extensions introduce more realism into the model, they increment its analytical complexity. For an extensive review of FPM see Bielecki and Rutkowski (2002, Chapter 3) and references therein.

The default threshold, always positive, can be interpreted in various ways. We can think of it as a safety covenant of the firm's debt which allows the bondholders to take control of the company once its asset value has reached this level. The safety covenant would act as a protection mechanism for the bondholders against an unsatisfactory

⁷See Iori (2003) and Jeanblanc and Rutkowski (2000, Chp. 3.1).

corporate performance. In this case, the default threshold would be deterministic, although possibly time dependent, and exogenously fixed when the firm's debt is issued. Kim, Ramaswamy and Sundaresan (1993) and Longstaff and Schwartz (1995) assume an exogenously given constant default threshold K . Black and Cox (1976) consider a time dependent default barrier given by $e^{-\gamma(T-t)}K$. A particular case of Black and Cox default threshold specification is to consider $\gamma = r$, i.e. to consider a default barrier equal to the face value of the debt discounted at the risk-free interest rate. In that case, the default threshold can be made stochastic if the model considers a stochastic process for the interest rate, as in Briys and de Varenne (1997).

Longstaff and Schwartz (1995) choose a constant default threshold and point out that (p. 793) "since it is the ratio of V_t to K , rather than the actual value of K , that plays the major role in our analysis, allowing a more general specification for K simply makes the model more complex without providing additional insight into the valuation of risky debt."

Hsu, Saá-Requejo and Santa-Clara (2004) suggest that V_t and K do not matter directly to the valuation of default risky bonds but only through their ratio, which is a measure of the solvency of the firm. They model the default threshold as a stochastic process, which together with the stochastic process assumed for the firm's asset value, allow them to obtain the stochastic process of the ratio $\frac{V_t}{K}$. The dynamics of the ratio $\frac{V_t}{K}$ are used to price corporate bonds.

The default threshold can also be chosen endogenously by the stockholders to maximize the value of the equity. See for example Mello and Parsons (1992), Nielsen et al. (1993), Leland (1994), Anderson and Sundaresan (1996), Leland and Toft (1996), Mella-Barral and Perraudin (1997), and François and Morellec (2004).

The literature has also considered the possibility of negotiation processes between stockholders and bondholders when the firm goes near the point of financial distress, from which the default threshold is determined.⁸

⁸The reader interested in the literature on strategic debt service may see Mella-Barral and Perraudin (1997), Fan and Sundaresan (2000) and references therein.

Similarly as how we described the choice of the face-value of the zero-coupon in the Merton model, in FPM the default threshold can be calculated as a weighted average of short and long-term debts.

Interest rates can be considered either as a constant or as a stochastic process.⁹ The stochasticity of interest rates allows the model to introduce correlation between asset value and interest rates, and to make the default threshold stochastic, in the cases it is specified as the discounted value of the face value of the debt. Nielsen et al. (1993) and Longstaff and Schwartz (1995) consider a Vasicek process for the interest rate, correlated with the firms' asset value:

$$dV_t = (c - d) V_t dt + \sigma_V V_t dW_t, \quad (13)$$

$$dr_t = (a - br_t) dt + \sigma_r d\bar{W}_t, \quad (14)$$

$$d\bar{W}_t dW_t = \rho dt, \quad (15)$$

where \bar{W}_t and W_t are correlated Brownian motions. Other specifications for the stochastic process of the short rate have been considered. For example Kim, Ramaswamy and Sundaresan (1993) suggest a CIR process

$$dr_t = (a - br_t) dt + \sigma_r \sqrt{r_t} d\bar{W}_t, \quad (16)$$

and Briys and de Varenne (1997) a generalized Vasicek process

$$dr_t = (a(t) - b(t) r_t) dt + \sigma_r(t) d\bar{W}_t. \quad (17)$$

Hsu, Saá-Requejo and Santa-Clara (2004) consider both the case of independence between risk-free interest rates and the default generating mechanism (given by the dynamics of the ratio $\frac{V_t}{K}$) and the case of correlation between both processes, specifying the risk-free rate as a CIR process. They present an interesting empirical illustration of the model, covering the calibration of the risk-free rate process and the estimation of the model's parameter through the Generalized Method of Moments.

⁹See Black and Cox (1976), Leland (1994), and Leland and Toft (1996) for models with constant interest rates, and see Kim, Ramaswamy and Sundaresan (1993), Nielsen et al. (1993), Longstaff and Schwartz (1995), Bryis and de Varenne (1997), Collin-Dufresne and Goldstein (2001), and Hsu, Saá-Requejo and Santa-Clara (2004) for models with stochastic interest processes.

2.2.1 Drawbacks and extensions

The principal drawback of FPM is the analytical complexity that they introduce, which is increased if we consider stochastic interest rates or endogenous default thresholds. This mathematical complexity makes difficult to obtain closed form expressions for the value of the firm's equity and debt, or even for the default probability, forcing us to make use of numerical procedures.

The empirical testing of FPM and structural models in general has not been very successful.¹⁰ Eom, Helwege and Huang (2003), who carry out an empirical analysis of five models (Merton, Geske, Leland and Toft, Longstaff and Schwartz, and Collin-Dufresne and Goldstein), conclude that (p. 502)

“Using estimates from the implementations we consider most realistic, we agree that the five structural bond pricing models do not accurately price corporate bonds. However, the difficulties are not limited to the underprediction of spreads. ... they all share the same problem of inaccuracy, as each has a dramatic dispersion of predicted spreads.”

Zhou (1997, Abstract) indicates that “the empirical application of a diffusion approach has yielded very disappointing results.”

Another drawback of the structural models presented before is the so-called *predictability of defaults*. Generally, structural models consider continuous diffusion processes for the firm's asset value and complete information about asset value and default threshold. In this setting, the actual distance from the asset value to the default threshold tells us the nearness of default, in such a way that if we are far away from default the probability of default in the short-term is close to zero, because the asset value process needs time to reach the default point. The knowledge of the distance of default and the fact that the asset value follows a continuous diffusion process makes default a predictable event, i.e. default does not come as a surprise.

¹⁰See Anderson and Sundaresan (2000), Eom, Helwege, Huang (2003), and Ericsson and Reneby (2004).

This predictability of defaults makes the models generate short-term credit spreads close to zero. In contrast, it is observed in the market that even short-term credit spreads are bounded from below, incorporating the possibility of an unexpected default or deterioration in the firm's credit quality.¹¹

The same characteristics of the structural models that imply the predictability of default also imply *predictability of recovery*. In models which do not consider strategic defaults, the bondholders get the remaining value of the firm in case of default, which is precisely the value of the default threshold at default. Thus, if we assume complete information about asset value and default threshold, the recovery rate is also a predictable quantity.

Essentially, two ways out of this predictability effects of structural models have been proposed in the literature. The predictability of default comes from the assumption of investors' perfect knowledge of the firm's asset value and default threshold. In practice, it is not possible to deduce from the capital structure of the firm neither the value of the firm V_t , its volatility σ_V , nor the level of the default threshold. If we consider incomplete information about either the firm value process, the default threshold (or both), investors can only infer a distribution function for these processes, which makes defaults impossible to predict. These considerations can be found, among others, in Duffie and Lando (2001), Giesecke (2005) and Jarrow and Protter (2004).¹²

The second way consists in incorporating jumps in the dynamics of the firm value, which implies that the asset value of the firm can suddenly drop, reducing drastically the distance of default (between asset value and default threshold), or even causing a default if the drop is sufficiently high. Thus, default is not a predictable event any more, the default probabilities for short maturities do not tend to zero and so

¹¹See Jones et al. (1984), Franks and Torous (1989), Sarig and Warga (1989), Fons (1994), Huang and Huang (2003), and Leland (2004).

¹²Elizalde (2005b) presents a review of structural models appeared in the literature which consider incomplete information assumptions and bridge the gap between the structural and the reduced approach.

the credit spreads generated. Zhou (1997, 2001a) and Hilberink and Rogers (2002) deal with structural models in which the firm's asset value incorporates a jump component. While Zhou extends Longstaff and Schwartz (1995) model considering a lognormally distributed jump component, Hilberink and Rogers (2002) opt for an extension of Leland (1994) and Leland and Toft (1996) using Levy processes which only allow for downward jumps in the firm's value. Both models avoid the problem of default predictability implying positive credit spreads for short maturities. Another characteristic of jump models is that they convert the recovery payment at default in a random variable, since the value of the firm can drop suddenly below the default threshold, whereas if the firm's value follows a diffusion process without jumps, the value of the firm at default, i.e. what bondholders get is always equal to the default threshold because of the continuity of the firm's value path.

Fouque, Sircar and Solna (2005) consider the effect of introducing stochastic volatility in FPM, finding that it increases short-term spreads.

Davydenko (2005) criticizes existing structural models because they obviate liquidity reasons as the main determinants of default for some firms, particularly the ones with high external financing costs (p. 2):

“Several default triggers have been proposed in structural models of debt pricing. Most models assume that a firm defaults when the market value of its assets falls below a certain boundary (Black and Cox, 1976; Leland, 1994). This default boundary may correspond to an exogenous net-worth covenant, or to the endogenously determined threshold at which equityholders are no longer willing to service debt obligations. Should the firm find itself in a liquidity crisis while its asset value is still above the boundary, equityholders in these models will always be willing and able to avoid default by raising outside financing. This approach contrasts with the assumption that firms default when current assets fall short of current obligations, due to either a minimum cash-flow covenant, or market

frictions precluding the firm from raising sufficient new external financing (Kim et al., 1993; Anderson and Sundaresan, 1996). Models incorporating both value- and liquidity-based defaults are rare, and little empirical evidence is available to motivate the choice of the default trigger. If, in reality, default is triggered by different factors for different firms, existing models are likely to lack accuracy in predictions.”

Davydenko (2005), using a sample of US (speculative rating-grade) bond issuers from 1996 to 2003, shows that the importance of liquidity shortages in triggering default for a particular firm depends on the firm’s cost of external financing (p. 2): “firms with low costs of external financing default when the continuation value of assets is low. By contrast, if external funds are costly, a liquidity crisis may force reorganization even if the going-concern surplus is still substantial.”¹³ Moreover, the author presents empirical evidence against the view that default is triggered when the asset value crosses a particular threshold.

Therefore, empirical evidence suggests that structural models need to be theoretically extended in order to incorporate the possibility of the firms defaulting because of liquidity shortages and high funding costs.

2.3 Estimation

The literature provides several ways of calibrating V_t and σ_V . The first method makes use of Itô’s Lemma to obtain a system of two equations in which the only two unknown variables are V_t and σ_V .¹⁴ Assume the firm’s equity value follows a geometric Brownian motion under \mathbf{P} , with volatility σ_E :

$$dE_t = rE_t dt + \sigma_E E_t dW_t. \quad (18)$$

¹³At any given point in time the firms’ owners face a decision of whether to liquidate the firm, or to maintain the status quo by continuing operations under the current regime, also referred to a going concern.

¹⁴See, for example, Jones et al. (1984), Ronn and Verma (1986), Eom et al. (2000), Delianedis and Geske (2003), and Ericsson and Reneby (2005).

Since the value of the equity is a function of time and of the value of the assets, $E_t = f(V_t, t)$, we can apply Itô's Lemma to get

$$dE_t = \left[\frac{\delta f(V_t, t)}{\delta t} + \frac{\delta f(V_t, t)}{\delta V_t} V_t r + \frac{1}{2} \frac{\delta^2 f(V_t, t)}{(\delta V_t)^2} (V_t \sigma_V)^2 \right] dt + \frac{\delta f(V_t, t)}{\delta V_t} V_t \sigma_V dW_t. \quad (19)$$

Comparing the coefficients multiplying the Brownian motion in the two previous equations we obtain the following identity

$$\sigma_E E_t = \frac{\delta f(V_t, t)}{\delta V_t} V_t \sigma_V \quad (20)$$

Noting that $\frac{\delta f(V_t, t)}{\delta V_t} = \frac{\delta E_t}{\delta V_t} = \Phi(d_1)$ and rearranging we obtain the first equation of the system:¹⁵

$$\sigma_V = \frac{E_t}{V_t} \sigma_E \Phi(d_1). \quad (21)$$

The second equation results simply from matching the theoretical value of equity with the observed market price (\hat{E}_t):

$$E_t(V_t, \sigma_V, T - t) = \hat{E}_t. \quad (22)$$

As we mentioned before, the only two unknowns in the system formed by the last two equations are V_t and σ_V .¹⁶

Duan (1994) points out some drawbacks of the previous method. First, the method considers the equity volatility as constant and independent of the firm's asset value and time. Second, he claims that the first equation is redundant since it is used to derive the second equation. And third, the traditional method does not provide us with distribution functions, or even confidence intervals, for the estimates of V_t and σ_V .

Duan (1994) proposes another method of estimating V_t and σ_V , based on maximum likelihood estimation using equity prices and the one-to-one relationship be-

¹⁵Crosbie and Bohn (2003) point out that this equation holds only instantaneously, and that in practice, market leverage, which would be represented here by $\frac{e^{-t(T-t)}D}{V_t}$, moves around far too much for that equation to provide reasonable results.

¹⁶Ronn and Verma (1986) extend the estimation to the cases of non-stationary σ_E and stochastic interest rates.

tween equity and asset levels given by (4).¹⁷ Duan et al. (2003) follow the maximum likelihood approach introduced by Duan (1994) but, unlike previous works, they take into account the survivorship issue, by incorporating into the likelihood function the fact that a firm survived. They argue that (p. 3) “In the credit risk setting, it is imperative for analysts to recognize the fact that a firm in operation has by definition survived so far. Estimating a credit risk model using the sample of equity prices needs to reflect this reality, or runs the risk of biasing the estimator.”

Duan and Fulop (2005) extend Duan’s (1994) maximum likelihood estimation method to account for the fact that observed equity prices might be contaminated by trading noises. They find that taking into account trading noises generates lower estimates for the asset volatility σ_V and therefore overestimates the firms’ default probabilities.

Bruche (2005) describes how structural models can be estimated using a simulated maximum likelihood procedure, which allows to use data on any of the firm’s traded claims (bonds, equity, CDS, ...) as well as balance sheet information to improve the efficiency of the estimation. The paper explores the possibility of considering that not only equity, but the rest of the claims used in the estimation procedure can be priced with noise, showing that (p. 3) “even small amounts of noise can have serious consequences for estimation results when they are ignored.”

A different way of estimating V_t and σ_V , which can be found on Jones et al. (1984), consists simply on estimating the asset value as the sum of the equity market value, the market value of traded debt and the estimated value of non-traded debt. Provided with a time series for V_t we can estimate its volatility σ_V .

Hull, Nelken and White (2004) propose a way to estimate the model’s parameters from implied volatilities of options on the company’s equity, avoiding to estimate σ_E and to transform the firm’s debt structure into a zero-coupon bond. Using as inputs two equity implied volatilities and an estimate of the firm’s debt maturity T , their

¹⁷For a complete description of this method see Duan (1994), Duan et al. (2002) and Ericsson and Reneby (2005) - who also present a comparison with the traditional method.

model provides us with an estimate of σ_V and the leverage ratio $\frac{De^{-r(T-t)}}{V_t}$, which allows us to calculate E_t and the probability of default. We should note that to calculate the value of the debt $z(t, T) = V_t - E_t$ we still need an estimate for V_t .¹⁸

We still have to estimate the default threshold K . Sundaram (2001) indicates that (p. 7) “default tends to occur in practice when the market value of the firm’s assets drops below a critical point that typically lies below the book value of all liabilities, but above the book value of short-term liabilities.” Thus, one approach is to choose a value for D between those two limits. Davydenko (2005) estimates the default threshold to be around 72% of the firm’s face value of debt.

2.4 Liquidation process models

In FPM default occurs the first time the asset value goes below a certain lower threshold, i.e. the firm is liquidated immediately after the default event; the default event corresponding to the crossing of the asset value through the lower barrier. In contrast with FPM, a new set of models has emerged where the default event does not immediately cause liquidation but it represents the beginning of a process, the liquidation process, which might or might not cause liquidation after it is completed. We label those models Liquidation Process Models LPM.

The distinction between the terms *default event* and *liquidation* must be clear to understand LPM and their differences with FPM. A default event takes place when the firm’s asset value V_t goes below the lower threshold K (which can be exogenous, constant, time dependent, stochastic or endogenously derived). A default event signals the beginning of a financially distressed period, which will not necessarily lead to liquidation. Liquidation takes place when the firm is actually liquidated, its activity stops and its remainings distributed among its claimholders.

In FPM described above the default event does coincide with liquidation.¹⁹ How-

¹⁸Eom et al. (2000) suggest another procedure to estimate σ_V , which they term *bond-implied volatility method*.

¹⁹In fact, we called simply default to such an occurrence. In what follows we shall use the terms default and liquidation with the same meaning (different than default event!). A default event starts

ever, as pointed out by Couderc and Renault (2005, p. 2), most liquidations “do not arise suddenly but are rather the conclusion of a long lasting process.” As pointed out by Moraux (2004, p. 3): “Empirical studies in USA have found that additional ‘survival’ periods beyond the main default event last up to 3 years (Altman-Eberhart (1994), Betker (1995), Hotchkiss (1995)). Helwege (1999) reports that the longest default of modern US junk bond market is seven years long.”²⁰

The fact that the liquidation process can take quite a while implies that when empirically studying the causes of liquidation past information shows up as a significant explanatory variable, together, of course, with contemporaneous information, because it comprises information about the liquidation process. Information here referring to the firms’ financial variables as well as financial markets, business cycle, credit markets and default cycle indicators. Couderc and Renault (2005) use a database containing the rating history of over ten thousand firms for the period 1981-2003 and analyze, using duration models, whether past values of several financial markets (business cycle, credit markets and default cycle) are relevant in explaining default probabilities in addition to their contemporaneous values. Their results show the critical importance of past information in default probabilities.

LPM extend FPM to account for the fact that the liquidation time takes place after (sometimes quite a lot after) the occurrence of a default event. François and Morellec (2004), Moraux (2004), and Galai, Raviv and Wiener (2005) put forward (the first to the best of our knowledge) theoretical LPM.

François and Morellec (2004) argue that while in most of FPM the default event leads to an immediate liquidation of the firm’s assets, firms in financial distress have several options to deal with their distress. First, under Chapter 7 of the US Bankruptcy Code, they can liquidate its assets straight away. This possibility would fit FPM. However firms can also file for bankruptcy under Chapter 11 of the US Bankruptcy Code. The process of liquidation has two possible endings: liquidation or default and reorganization (which happens when the firm manages to improve its financial health and avoid closure).

²⁰See also Frank and Torous (1989) and Gilson (1997).

ruptcy Code and start a court-supervised liquidation process. The authors refer to existing literature (p. 390) to provide some evidence about the relevance of Chapter 11 :

“Upon default, the court grants the firm a period of observation during which the firm can renegotiate its claims. At the end of this period, the court decides whether the firm continues as a going concern or not.

Empirical studies show that most firms emerge from Chapter 11. Only a few firms (5%, according to Gilson, John, and Lang [1990] and Weiss [1990], and between 15% and 25%, according to Morse and Shaw [1988]) are eventually liquidated under Chapter 7 after filing Chapter 11. Why do some firms recover while others do not? It is generally acknowledged (see Wruck 1990 or White 1996) that there exist two types of defaulting firms. First, firms that economically sound promptly recover under Chapter 11. Default was only due to a temporary financial distress. Second, firms that are economically unsound keep on losing value under Chapter 11.”²¹

François and Morellec consider that, after a default event, i.e. after the asset value V_t goes below the lower threshold K , a firm is liquidated if and only if V_t remains below K *consecutively* during a period of time of a given length d (which in their numerical simulations they take to be 2 years). If a default event happens and the asset value remains under the lower threshold for a period lower than d the liquidation process finishes and the firm continues in business as usual. The term *consecutively* in the definition of liquidation above means that the number of successfully managed past default events and liquidation periods²² the firms has experienced does not affect the maximum length d of future liquidation periods.

The authors provide closed-form solutions for corporate debt and equity values and analyze the implications of the model for optimal leverage and credit spreads.

²¹See also Kahl (2001) and Morrison (2003).

²²*Successfully managed* means that such liquidation periods did not last longer than d and, as a consequence, did not trigger liquidation.

Numerical simulations show that credit spreads are an increasing function of the length d .

Moraux (2004) extends François and Morellec (2004) model including an additional cause of liquidation to François and Morellec's one (which they call liquidation procedure A). Under his proposed liquidation procedure, procedure B, liquidation happens when the total, i.e. *cumulative*, time the firm assets value stands under the default threshold exceeds d . The difference between procedures A and B lies in the words *consecutively* and *cumulative*, and Moraux (2004, p. 17) explains it clearly:

“Under the procedure A, each time the firm value process passes through and above K , the liquidation procedure is closed and the hypothetical distress counter is set to zero. The next time a default event occurs, an identical procedure is run and an equal period of time d is granted. ... Under the procedure B, the distress counter is never set to zero. Subsequent granted periods (and therefore tolerance) will be lower and lower as more default events and long financial distress will be observed. In fact, the granted time is lowered (each time) by the duration just used.”

Financial distress refers to the situation in which $V_t < K$. A firm can be liquidated by either one or the other liquidation procedures. Moraux (2004) shows that any liquidation procedure based on the time spent by the firm in financial distress is bounded by the procedures A and B in the sense that its implied liquidation date will be higher (lower) than the liquidation date implied by procedure B (A).

The author derives closed form solutions for different claims such as equity, different seniority debts and convertible debt. In particular, the value of equity is derived as a Down and Out Parisian option written on the firm assets under liquidation procedure A and as a Down and Out cumulative call option under liquidation procedure B. Numerical simulations show that the value of equity is an increasing function of d , and that, unlike in François and Morellec (2004), credit spreads increase or decrease with d depending on the seniority of the debt.

Galai, Raviv and Wiener (2005) represent an step forward in the refinement of LPM, proposing a model extending and including the two previous ones. Galai, Raviv and Wiener argue that in the two previous models, the only thing that matters for a firm to be liquidated is the amount of time it spends in financial distress (either successively or cumulatively), but they fail to (p. 5) “capture the following two common features of bankruptcy procedures: (i) Recent distress events may have a greater effect on the decision to liquidate a firm’s assets than old distress events. ... (ii) Severe distress events may have greater effect on the decision to liquidate a firm than mild distress events.” To account for such two stylized facts, the authors propose a structural model in which a firm is liquidated when a state variable representing the *cumulative weighted* time period spent by the firm in distress exceeds d . At each time, the cumulative weighted time period is computed as a weighted average of the total time spent by the firm in distress, weighted by (i) how far away in the past such distress occurred and (ii) how severe was such a distress, where distress severity is measured as an increasing function of $\max\{0, K - V\}$.

Galai, Raviv and Wiener’s model have as special cases models such as Merton (1974), Black and Cox (1976), Leland (1994), Fan and Sundaresan (2001), François and Morellec (2004) and Moraux (2004). As a consequence it represents the more general LPM so far. The authors solve the model numerically using Monte Carlo simulation based on Parisian options and Parisian contracts techniques to value debt and equity. They provide a very intuitive comparison of the liquidation mechanics in their general model with François and Morellec’s and Moraux’s ones, showing that Moraux’s cumulative liquidation procedure (B) have too strong memory because far away distress periods have the same impact on liquidation triggering than current ones.

Although theoretically very appealing, LPM have not, unlike FPM, been empirically tested, which remains a field for future research.

2.5 State dependent models

Another avenue for (so far) theoretical research within the structural approach consists of extending standard models with regime switching: some of the model parameters are state-contingent. As we review below, states can represent the state of the business cycle or simply the firm's external rating. Cash-flows, bankruptcy costs and funding costs might be state-dependent.

This branch of structural models is able to reduce the problems of predictability of defaults (and recovery) suffered by standard models because the firm is subject to exogenous changes of parameters which affect its ability to generate cash flows or its funding costs, which are the main drivers of default probabilities.

Hackbarth, Miao and Morellec (2004) and Elizalde (2005c) put forward two different models illustrating the previous ideas. In both cases the authors provide closed form expressions for the value of equity and debt, whose solution imply solving systems of ordinary differential equations.

In Hackbarth, Miao and Morellec (2004) cash flows and recovery rates depend on the state of the business cycle. Cash flows x_t follow a geometric Brownian motion and are scaled by a business cycle scalar factor: they are higher in expansions $y_H x_t$ than in recessions $y_L x_t$, $y_H > y_L$. In the same way, bankruptcy costs are expressed as a state-dependent fraction $1 - \alpha$ of the firm's assets; again, the recovery rate in expansions α_H is higher than in recessions α_L , $\alpha_H > \alpha_L$. At each point in time, there is an exogenous probability of switching between recession and expansion. The default threshold is endogenously chosen by equityholders to maximize the value of equity, and it turns out to be higher in recessions: the firm defaults earlier in recessions than in expansions. Numerical examples illustrate the implications of the model for default thresholds, default clustering, optimal leverage (countercyclical) and credit spreads. As argued above the model is able to generate non-trivial short term spreads.

Elizalde (2005c) develops a structural model which, although originally applied to banks, can be extended to any firm (in particular, Section 4.3). In contrast with

previous models, the firms' asset value is assumed to be unobserved by debtholders. Debtholders rely on the ratings published by rating agencies to set the debt's coupon as a function of those ratings. As a consequence, the firms' funding costs are contingent on their ratings. Rating agencies perform timely audits to firms, with a given frequency, to find out their risk and asset levels, which determine the rating. Switching from one rating to another implies changes in the cost of debt and, as a consequence, in the ability of the firm to repay it. As in Hackbarth, Miao and Morellec (2004) the default threshold is chosen endogenously by equityholders and it is rating-dependent.

As described by Duffie (2005, p. 2772), "It has become increasingly common for bond issuers to link the size of the coupon rate on their debt with their credit rating, offering a higher coupon rate at lower ratings, perhaps in an attempt to appeal to investors based on some degree of hedging against a decline in credit quality. This embedded derivative is called a '*ratings-based step-up*.'" The author illustrates an example of a ratings-based step-up bond issued by Deutsche Telecom in 2002 with coupon payments linked to the firm's rating. While Elizalde (2005c) derives the price of such a bond using a structural model, Duffie provides its pricing formula using an intensity model.²³

Like LPM, State Dependent Models (SDM) have only been developed theoretically and their future success in credit risk modelling (if any) lies in their empirical applicability and their ability to replicate and predict credit spreads and default probabilities.

3 Default correlation

To incorporate default dependencies between firms using structural models the literature has essentially relied on natural extensions of single firm's models, either Merton (1974) type models or FPM. We will start this section reviewing these extensions, un-

²³Manso, Strulovici and Tchisty (2004) present an alternative derivation of ratings-based step up bonds using structural models, and review the existing literature.

der which the default dependences between firms are introduced through correlated asset processes.²⁴ Giesecke (2004) and Giesecke and Goldberg (2004) suggest that the default correlation implied by the use of correlated firms' asset processes accounts for the dependence of the firms' credit quality on common macroeconomic factors, what they term *cyclical default correlation*, but it does not account for credit risk contagion across firms and periods of default clustering. In order to introduce the *contagion correlation* in the model, Giesecke (2004) and Giesecke and Goldberg (2004) propose a model in which the firms' default thresholds are dependent one to each other and are unknown to investors.

After reviewing Giesecke (2004) and Giesecke and Goldberg (2004) we present Factor Models, which express the value of the firms' assets as a function of several common factors, which generate the correlation, and an idiosyncratic factor.²⁵ Duan et al. (2002) and Hull and White (2001) present two alternative approaches to deal with default correlation in structural models.

3.1 Cyclical default correlation

The most natural way to introduce default dependences between firms in structural models is by correlating the firms' asset processes.²⁶ Suppose we have $i = 1, \dots, I$ different firms with asset value processes given by

$$dV_{i,t} = rV_{i,t}dt + \sigma_{V_i}V_{i,t}dW_{i,t}, \quad (23)$$

for $i = 1, \dots, I$, where $W_{1,t}, \dots, W_{I,t}$ are correlated Brownian motions. As in the single firm case, these models imply predictable defaults. One way of getting rid of the default predictability would be to introduce jump components in the firms' asset processes. Those jump components could be either correlated or uncorrelated across firms. Correlated jump components, besides making defaults unpredictable, would

²⁴See Zhou (2001b).

²⁵See Schönbucher (2000), Finger (1999), and Frey, McNeil and Nyfeler (2001).

²⁶See Zhou (2001b).

also account for credit risk contagion effects. The main problem lies in the calibration of those jump components.

3.2 Contagion default correlation

Cyclical default correlation does not account for all the credit risk dependence between firms. Giesecke (2004) and Giesecke and Goldberg (2004) extend structural models for default correlation to incorporate credit risk contagion effects. The default of one firm can trigger the default of related firms. Furthermore, default times tend to concentrate in some periods of time in which the probability of default of all firms is increased and which can not be totally, or even partially, explained by the firms' common dependence on some macroeconomic factors.

Contagion effects can arise in this setting by direct links between firms in terms of, for example, commercial or financial relationships. The news about the default of one firm have a big impact in the credit quality of other related firms, which is immediately reflected in their default probabilities.

In structural FPM we assume that investors have complete information about both asset processes and default thresholds, so they always know the nearness of default for each firm, i.e. the distance between the actual level of the firm's assets and its default threshold.²⁷

Giesecke (2004) and Giesecke and Goldberg (2004) introduce contagion effects in the model by relaxing the assumption that investors have complete information about the default thresholds of the firms. In Giesecke (2004), bond holders do not have perfect information about neither about such thresholds nor about their joint distribution. However, they form a prior distribution which is updated at any time one of such thresholds is revealed, which only happens when the corresponding firm defaults. In Giesecke (2004) investors have incomplete information about the firms' default thresholds but complete information about their asset processes. Giesecke and Goldberg (2005) extend that framework to one in which investors do not have

²⁷We are not considering here jump components in the dynamics of the assets processes.

information neither about the firms' asset values nor about their default thresholds. In this case, default correlation is introduced through correlated asset processes and, again, investors receive information about the firms' asset and default barrier only when they default. Such information is used to update their priors about the distribution of the remaining firms' asset values and default thresholds.

The incomplete information about the level of the default thresholds and the fact that those levels are dependent among firms (through a copula function) generate the source of credit risk contagion. Investors form a belief about the level of the firms' default thresholds. Each time one of the firms defaults, the true level of its default threshold is revealed, and investors use this new information to update their beliefs about the default thresholds of the rest of the firms. This sudden updating of the investors' perceptions about the default thresholds of the firm, and thus about the nearness of default for each firm, introduces the default contagion effects in the models.

This model allows the introduction of default correlation both through dependences between firms' asset values, cyclical default correlations, and through dependences between firms' default barriers, contagion effects.

The major problem of this approach is to calibrate and estimate the default threshold copula. See Giesecke (2003) for some remarks on how to choose and calibrate that copula.

3.3 Factor models

Factor models consider the firms' asset values as a function of a group of common factors, which introduce the default correlation in the model, plus a firm's specific factor:

$$V_{i,t} = \sum_{j=1}^J w_{i,j} Z_{j,t} + \epsilon_{i,t}, \quad (24)$$

where Z_1, \dots, Z_J represent the common factors, $\epsilon_1, \dots, \epsilon_I$ the firms' specific factors (independent of Z_1, \dots, Z_J), and the correlation structure is given by the coefficients w .

Once we know the realization of the common factors, the firms' asset value and thus the firms' default probabilities are independent.

The calibration of factor models is usually carried out by a logit or probit regression, depending on the assumptions about the distribution of the factors. Schönbucher (2000), Finger (1999), and Frey, McNeil and Nyfeler (2001) present illustrations of these models.

References

- [1] Acharya, V., and Carpenter, J., 2002, "Corporate Bond Valuation and Hedging with Stochastic Interest Rates and Endogenous Bankruptcy," *Review of Financial Studies* 15, 1355-1383.
- [2] Acharya, V., Huang, J. Z., Subrahmanyam, M., and Sundaram, R., 2000, "Costly Financing, Optimal Payout Policies and the Valuation of Corporate Debt," Working Paper, NYU.
- [3] Altman, E., and Eberhart, A., 1994, "Do priority provisions protect a bond-holder's investment?," *Journal of Portfolio Management* 20, 67-75.
- [4] Anderson, R., and Sundaresan, S., 1996, "Design and Valuation of Debt Contracts," *Review of Financial Studies* 9, 37-68.
- [5] Anderson, R., and Sundaresan, S., 2000, "A Comparative Study of Structural Models of Corporate Bond Yields," *Journal of Banking and Finance* 24, 255-269.
- [6] Anderson, R., Sundaresan, S., and Tychon, P., 1996, "Strategic Analysis of Contingent Claims," *European Economic Review* 40, 871-881.
- [7] Betker, B., 1995, "An empirical examination of prepackaged bankruptcy," *Financial Management* 24, 3-18.
- [8] Bielecki, T. R., and Rutkowski, M., 2002, "Credit Risk: Modeling, Valuation and Hedging," *Springer Finance*.
- [9] Bingham, N. H., and Kiesel, R., 2002, "Semi-parametric Modelling in Finance: Theoretical Foundations," *Quantitative Finance* 2, 241-250.
- [10] Black, F., and J. C. Cox, 1976 "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions," *Journal of Finance* 31, 351-367.
- [11] Black, F., and Scholes, M., 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81, 637-654.
- [12] Briys, E., and de Varenne, F., 1997, "Valuing Risky Fixed Rate Debt: An Extension," *Journal of Financial and Quantitative Analysis* 31, 239-248.
- [13] Bruche, M., 2005, "Estimating structural bond pricing models via simulated maximum likelihood," London School of Economics, Financial Markets Group Discussion Paper 534.
- [14] Collin-Dufresne, P., and Goldstein, R., 2001, "Do Credit Spreads Reflect Stationary Leverage Ratios?," *Journal of Finance* 56, 1929-1957.
- [15] Couderc, F., and Renault, O., 2005, "Times-to-Default: Life Cycle, Global and Industry Cycle Impacts," FAME Research Paper No. 142.
- [16] Crosbie, P. J., and Bohn, J. R., 2002, "Modelling Default Risk," Working Paper, KMV Corporation.
- [17] Davydenko, S., 2005, "When do firms default?," University of Toronto Working Paper.

- [18] Delianedis, G., and Geske, R., 2003, "Credit Risk and Risk Neutral Default Probabilities: Information about Rating Migrations and Defaults," UCLA Working Paper.
- [19] Delianedis, G., and Geske, R., 2001, "The Components of Corporate Credit Spreads: Default, Recovery, Tax, Jumps, Liquidity, and Market Factors," Working Paper, UCLA.
- [20] Duan, J. C., 1994, "Maximum Likelihood Estimation Using Price Data of the Derivative Contract," *Mathematical Finance* 4, 155-157.
- [21] Duan, J. C., and Fulop, A., 2005, "Estimating the structural credit risk model when equity prices are contaminated by trading noises," University of Toronto Working Paper.
- [22] Duan, J. C., Gauthier, G., Simonato, J. G., and Zaanoun, S., 2002, "Maximum Likelihood Estimation of Structural Credit Spread Models: Deterministic and Stochastic Interest Rates," Working Paper.
- [23] Duan, J. C., Gauthier, G., Simonato, J. G., and Zaanoun, S., 2003, "Estimating Merton's Model by Maximum Likelihood with Survivorship Consideration."
- [24] Duffie, D., 2005, "Credit risk modelling with affine processes," *Journal of Banking and Finance* 29, 2751-2802.
- [25] Duffie, D., and Lando, D., 2001, "Term Structure of Credit Spreads with Incomplete Accounting Information," *Econometrica* 69, 633-664.
- [26] Duffie, D., and Singleton, K. J., 2003, "Credit Risk: Pricing, Measurement and Management," *Princeton Series in Finance*.
- [27] Elizalde, A., 2005a, "Credit Risk Models I: Default Correlation in Intensity Models," available at www.abelelizalde.com.
- [28] Elizalde, A., 2005b, "Credit Risk Models III: Reconciliation Structural - Reduced Models," available at www.abelelizalde.com.
- [29] Elizalde, A., 2005c, "From Basel I to Basel II: An Analysis of the Three Pillars," available at www.abelelizalde.com.
- [30] Eom, Y. H., Helwege, J., and Huang, J. Z., 2003, "Structural Models of Corporate Bond Pricing: An Empirical Analysis," *Review of Financial Studies* 17, 499-544.
- [31] Ericsson, J., and Reneby, J., 2005, "Estimating Structural Bond Pricing Models," *Journal of Business* 78, 707-736.
- [32] Fan, H., and Sundaresan, S., 2000, "Debt Valuation, Renegotiation and Optimal Dividend Policy," *Review of Financial Studies* 13, 1057-1099.
- [33] Finger, C., 1999, "Conditional Approaches for CreditMetrics Portfolio Distributions," CreditMetrics Monitor, Spring, 14-26.
- [34] Fons, J. S., 1994, "Using Default Rates to Model the Term Structure of Credit Risk," *Financial Analysts Journal*.

- [35] Fouque, J., Sircar, R., and Solna, K., 2005, “Stochastic Volatility Effects on Defaultable Bonds,” *Applied Mathematical Finance*, forthcoming.
- [36] François, P., and Morellec, E., 2004, “Capital Structure and Asset Prices: Some Effects of Bankruptcy Procedures,” *Journal of Business* 77, 387-411.
- [37] Franks, J., and Torous, W., 1989, “An Empirical Investigation of U.S. Firms in Reorganization,” *Journal of Finance* 44, 747-769.
- [38] Frey, R., McNeil, A. J., and Nyfeler, M. A., 2001, “Modelling Dependent Defaults: Asset Correlations Are Not Enough!,” Working Paper, Department of Mathematics, ETHZ, Zurich.
- [39] Galai, D., Raviv, A., and Wiener, Z., 2005, “Liquidation Triggers and The Valuation of Equity and Debt”, Working Paper, School of Business Administration, The Hebrew University of Jerusalem.
- [40] Garbade, K., 1999, “Managerial Discretion and the Contingent Valuation of Corporate Securities,” *Journal of Derivatives* 6, 65-76.
- [41] Geske, R., 1977, “The Valuation of Corporate Liabilities as Compound Options,” *Journal of Financial and Quantitative Analysis* 12, 541-552.
- [42] Geske, R., 1979, “The Valuation of Compound Options,” *Journal of Financial Economics* 7, 63-81.
- [43] Gilson, S., John, K., and Lang, L., 1990, “Troubled Debt Restructurings: An Empirical Study of Private Reorganization of Firms in Default,” *Journal of Financial Economics* 27, 315-353.
- [44] Gilson, S., 1997, “Transaction Costs and Capital Structure Choice: Evidence from Financially Distressed Firms,” *Journal of Finance* 52, 161-196.
- [45] Goldstein, R., Ju, N., and Leland, H., 2001, “An EBIT-Based Model of Dynamic Capital Structure,” *Journal of Business* 74, 483-512.
- [46] Giesecke, K., 2004, “Correlated default with incomplete information,” *Journal of Banking and Finance* 28, 1521-1545.
- [47] Giesecke, K., 2005, “Default and Information,” Working Paper, Cornell University.
- [48] Giesecke, K., and Goldberg, L. R., 2004, “Sequential defaults and incomplete information,” *Journal of Risk* 7, 1-26.
- [49] Hackbarth, D., Miao, J., and Morellec, E., 2004, “Capital Structure, Credit Risk, and Macroeconomic Conditions,” FAME Research Paper No. 125.
- [50] Helwege, J., 1999, “How long do junk bonds spend in default?,” *Journal of Finance* 54, 341-357.
- [51] Hilberink, B., and Rogers, L. C. G., 2002, “Optimal Capital Structure and Endogenous Default,” *Finance and Stochastics* 6, 237-263.
- [52] Ho, T., and Singer, R., 1982, “Bond Indenture Provisions and the Risk of Corporate Debt,” *Journal of Financial Economics* 10, 375-406.

- [53] Hotchkiss, E., 1995, "Postbankruptcy performance and management turnover," *Journal of Finance* 50, 3-21.
- [54] Hsu, J., Saá-Requejo, J., and Santa-Clara, P., 2004, "Bond Pricing with Default Risk," UCLA Working Paper.
- [55] Huang, J. Z., 1997, "Default Risk, Renegotiation, and the Valuation of Corporate Claims," Ph.D. Dissertation, NYU.
- [56] Huang, J. Z., and Huang, M., 2003, "How Much of the Corporate-Treasury Yield Spread is Due to Credit Risk?," Working Paper, Penn State and Stanford Universities.
- [57] Hull, J., Nelken, I., and White, A., 2004, "Merton's Model, Credit Risk, and Volatility Skews," *Journal of Credit Risk* 1, 3-28.
- [58] Hull, J., and White, A., 2001, "Valuing Credit Default Swaps II: Modelling Default Correlations," *Journal of Derivatives* 8, 12-22.
- [59] Iori, G., 2003, "Exotic Options," Lecture Notes, MSc in Financial Mathematics, King's College London, available at www.giuliaiori.com.
- [60] Jarrow, R. A., and Protter, P., 2004, "Structural versus reduced form models: a new information based perspective," *Journal of Investment Management* 2, 1-10.
- [61] Jeanblanc, M., and Rutkowski, M., 2000, "Modelling of Default Risk: An Overview," *Mathematical Finance: Theory and Practice*, Higher Education Press, Beijing, 171-269.
- [62] Jones, P., Mason, S., and Rosenfeld, E., 1984, "Contingent Claim Analysis of Corporate Capital Structures: An Empirical Investigation," *Journal of Finance* 39, 611-625.
- [63] Kahl, M., 2001, "Financial Distress as a Selection Mechanism: Evidence from the United States," Anderson School, Finance Working Paper No. 16-01.
- [64] Kim, I. J., Ramaswamy, K., and Sundaresan, S. M., 1993, "Does default risk in coupons affect the valuation of corporate bonds?: A contingent claims model," *Financial Management* 22, 117-131.
- [65] Leland, H. E., 1994, "Risky Debt, Bond Covenants and Optimal Capital Structure," *Journal of Finance* 49, 1213-1252.
- [66] Leland, H. E., 1998, "Agency Costs, Risk Management, and Capital Structure," *Journal of Finance* 51, 987-1019.
- [67] Leland, H. E., and Toft, K. B., 1996, "Optimal Capital Structure, Endogenous Bankruptcy and the Term Structure of Credit Spreads," *Journal of Finance* 50, 789-819.
- [68] Leland, H. E., 2004, "Predictions of Default Probabilities in Structural Models of Debt," *Journal of Investment Management* 2.
- [69] Longstaff, F. A., and Schwartz, E. S., 1995, "A Simple Approach to Valuing Risky Fixed and Floating Rate Debt," *Journal of Finance* 50, 789-819.
- [70] Manso, G., Strulovici, B., and Tchisty, A., 2004, "Performance-Sensitive Debt," Working paper.

- [71] Mella-Barral, P., and Perraudin, W., 1997, "Strategic Debt Service," *Journal of Finance* 52, 531-566.
- [72] Mello, A., and Parsons, J., 1992, "Measuring the agency cost of debt," *Journal of Finance* 47, 1887-1904.
- [73] Merton, R., 1974, "On the Pricing of Corporate Debt: the Risk Structure of Interest Rates," *Journal of Finance* 29, 449-470.
- [74] Moraux, F., 2004, "Valuing Corporate Liabilities When the Default Threshold is not an Absorbing Barrier," University of Rennes Working Paper.
- [75] Morrison, E., 2003, "Bankruptcy Decision-Making: An Empirical Study of Small-Business Bankruptcies," Columbia Law and Economics Working Paper No. 239.
- [76] Morse, D., and Shaw, W., 1988, "Investing in Bankruptcy Firms," *Journal of Finance* 43, 1193-1206.
- [77] Nielsen, L. T., Saá-Requejo, J., and Santa-Clara, P., 1993, "Default Risk and Interest Rate Risk: The Term Structure of Default Spreads," Working Paper, INSEAD.
- [78] Ronn, E. I., and Verma, A. K., 1986, "Pricing Risk-Adjusted Deposit Insurance: An Option Based Model," *Journal of Finance* 41, 871-895.
- [79] Sarig, O., and Warga, A., 1989, "Some Empirical Estimates of the Risk Structure of Interest Rates," *Journal of Finance* 44, 1351-1360.
- [80] Schönbucher, P. J., 2000, "Factor Models for Portfolio Credit Risk," Working Paper, Department of Statistics, Bonn University.
- [81] Schönbucher, P. J., 2003, "Credit Derivatives Pricing Models," *Wiley Finance*.
- [82] Sundaram, R. K., 2001, "The Merton/KMV Approach to Pricing Credit Risk," NYU Working Paper.
- [83] Titman, S., and Torous, W., 1989, "Valuing Commercial Mortgages: An Empirical Investigation of the Contingent Claim Approach to Pricing Risky Debt," *Journal of Finance* 44, 345-373.
- [84] Weiss, L., 1990, "Bankruptcy Resolution: Direct Costs and Violation of Priority Claims," *Journal of Financial Economics* 27, 285-314.
- [85] White, M., 1996, "The Costs of Corporate Bankruptcy: A US-European comparison," included in "Corporate Bankruptcy: Economic and Legal Perspectives," Cambridge University Press.
- [86] Wruck, K., 1990, "Financial Distress, Reorganization, and Organizational Efficiency," *Journal of Financial Economics* 27, 419-444.
- [87] Zhou, C., 1997, "A Jump-Diffusion Approach to Modelling Credit Risk and Valuing Defaultable Securities," Federal Reserve Board, Washington.
- [88] Zhou, C., 2001a, "The Term Structure of Credit Spreads with Jump Risk," *Journal of Banking and Finance* 25, 2015-2040.
- [89] Zhou, C., 2001b, "An Analysis of Default Correlations and Multiple Defaults," *Review of Financial Studies* 14, 555-576.