Credit Risk Models III: Reconciliation Reduced - Structural Models

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Abstract

In recent years, some papers have tried to bridge the gap between the two main approaches in credit risk modelling: structural and reduced form models. Based on incomplete information versions of standard structural models, they are able to obtain reduced form models in which the intensity of default is not given exogenously but determined endogenously within the model and it is a function of the firm's characteristics and the level of information that investors posses.

The key element to link both approaches lies in the model's information assumptions. Using a specification of a structural model where investors do not have complete information about the dynamics of the processes which trigger the firm's default, these models derive a cumulative rate of default consistent with a reduced form model.

This paper pretends to be an introduction to this literature, providing some of the basic insights of the modelling structure and the main conclusions and results.

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Contents

This paper is part of a series of surveys on credit risk modelling and pricing. The complete list of surveys is available at `www.abelelizalde.com`, and consists on the following:

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5. Credit Default Swap Valuation: An Application to Spanish Firms.

"Structural models are based on the information set available to the firm's management, which includes continuous-time observations of both asset values and liabilities. Reduced-form models are based on the information set available to the market, typically including only partial observations of both the firm's asset values and liabilities.

[...] **it is possible to transform a structural model** with a predictable default time **into a reduced-form model**, with a totally inaccessible default time, **by altering the information sets available for modelling purposes**."

Guo, Jarrow and Zeng (2005, p. 2).

1 Introduction

In recent years, some papers have tried to bridge the gap between the two main approaches in credit risk modelling: structural and reduced form models. Based on incomplete information versions of standard structural models, they are able to obtain reduced form models in which the intensity of default is not given exogenously but determined endogenously within the model and it is function of the firm's characteristics and the level of information that investors posses.

The main distinguishing characteristic of structural models with respect to reduced form models is the link the former provide between the probability of default and the firms' fundamental financial variables: assets and liabilities. Reduced form models use market prices of the firms' defaultable instruments (such as bonds or credit default swaps) to extract both their default probabilities and their credit risk dependencies, relying on the market as the only source of information regarding the firms' credit risk structure. Although easier to calibrate, reduced form models lack the link between credit risk and the information regarding the firms' financial situation incorporated in their assets and liabilities structure.

The key element to link both approaches lies in the model's information assumptions. Using a specification of a structural model where investors do not have complete information about the dynamics of the processes which trigger the firm's default, these *reconciliation models* derive a cumulative rate of default consistent with a reduced form model.

Usually, in structural models, due to the assumption of complete information, investors are able to predict the arrival of default. This predictability of default implies zero short-term credit spreads for the firm's debt, which is not consistent with the short-term spreads seen in practice. Reduced form models overcome this limitation specifying an exogenous default intensity which makes default an unpredictable event. The main problem with reduced form models is that the arrival of default is not, *at least in modelling terms*, based on any characteristic of the firm's underlying credit quality.

Relaxing the complete information assumption, Duffie and Lando (2001), Çetin et al. (2004), Giesecke (2004, 2005) and Giesecke and Goldberg (2004a, b), and Guo, Jarrow and Zeng (2005), among others, arrive, through different although equivalent routes, to a framework which links both credit risk modelling approaches.

This report pretends to be an introduction to this literature, providing some of the basic insights of the modelling structure and the main conclusions and results. The reader interested in expanding the concepts provided here can make use of the reference list at the end of the text. Jarrow and Protter (2004) provide an alternative survey of the mentioned literature.

We start with an introduction of both structural and reduced approaches in Section 2. Section 3 reviews the concepts of compensator and pricing trend of a default process, which represent the methodological framework to connect both approaches. Finally, Section 4 presents a review of the main conclusions and results obtained in this literature.

2 Preliminaries

In what follows we shall, as background, always assume that economic uncertainty is modelled with the specification of a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$, where Ω is the set of possible states of nature, $\mathcal{F} = \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$, and $\mathbf{P}$ is a probability measure. The filtration $(\mathcal{F}_t)$ represents the information available to investors over time.

We consider two reference models, a structural model and an intensity-form model. As we shall become clear later, intensity and reduced form models are not exactly the same, although both terms have been used in the literature to refer to default risk models in which default is modelled as an exogenous event described by an intensity of arrival.[1] As a structural model, we will be referring to a first passage model.[2]

The structural form model consists on a specification for the firm's asset value V_t as a continuous process, and a specification for a default threshold process K_t such that the firm defaults the first time the asset value hits the default barrier, i.e. $\tau = \inf \{t \geq 0 \mid V_t \leq K_t\}$, where τ represents the time of default. To complete this structural model we need to specify what is known by the investors, i.e. the information structure $(\mathcal{F}_t)$. Traditionally, structural models assume complete information about asset and threshold processes, making the default time a predictable event.[3] The predictability of defaults implies that the short-rate credit spreads generated by the model tend to zero, whereas in practice we observe that they are bounded from below away from zero. The assumption of complete information about asset and default

[1]Intensity models represent the most extended type of reduced form models. Brody, Hughston and Macrina (2005) present an alternative reduced form model, based on the amount and precision of the information received by market participants about the firm's credit risk. Such model does not require the use of default intensities; it belongs to the reduced form approach because (as intensity models) it relies on market prices of defaultable instruments as the only source of information about the firms' credit risk.

[2]For a review of reduced and structural models see Elizalde (2005a, b).

[3]Giesecke and Goldberg (2004a, p. 11) consider that a stopping time is predictable when "the default event is foreshadowed by an observable phenomenon such as the value of the firm falling dangerously close to its default boundary". A more formal definition of predictable and unpredictable stopping times is given in Section 3, based on Schönbucher (2003, Section 4.1).

processes is not a realistic one from the point of view of investors, since it assumes that at each moment in time investors know the true value of the firm's assets and the true value of the default threshold. Relaxing the assumption of complete information makes the default time an unpredictable event. The reconciliation between structural and reduced form models is based on the information structure the models assumes, and different information frameworks will have different implications.

As Duffie and Singleton (2003, Section 5.5) point out, "with imperfect information, default occurs at some intensity, so one may view this structural model with imperfect information as formally equivalent to a reduced-form model that has the endogenously determined default intensity with first passage."

In contrast to classical models, the time of default in intensity models is not determined via the value of the firm, but it is the first jump of a point process. Intensity models assume the existence of an exogenously given process, the intensity of default, which represents the intensity of arrival of the default time. These models solve the problem of default predictability but they lack of an endogenous specification of default based on the firm's economic fundamentals.

A standard definition of the intensity of the default time τ, taken from Duffie and Lando (2001, p. 645-6), goes as follows: "The default stopping time τ has an intensity process λ_t with respect to the filtration $(\mathcal{F}_t)$ if λ_t is a non-negative progressively measurable process satisfying $\int_0^t \lambda_s ds < \infty$ a.s. for all t, such that

$$\left\{ \mathbf{1}_{\{\tau \leq t\}} - \int_0^t \lambda_s ds : t \geq 0 \right\} \tag{1}$$

is a $(\mathcal{F}_t)$-martingale."[4] λ_t is the conditional rate of default just after time t, given all information available up to that time, i.e.

$$\Pr\left(\tau \in (t, t+dt) \mid \mathcal{F}_t\right) = \lambda_t dt, \tag{2}$$

for $t \leq \tau$. And for each $T \geq t$,

$$\Pr\left[\tau \leq T \mid \mathcal{F}_t\right] = E\left[\exp\left\{-\int_t^T \lambda_s ds\right\} \mid \mathcal{F}_t\right], \tag{3}$$

[4] $\mathbf{1}_{\{.\}}$ represents the indicator function.

again for $t \leq \tau$.

The aim of the literature we are going to present is to link structural and reduced approach. It generally starts with an specification of an structural model with incomplete information about either the asset process, the default threshold or both, and through the use of the *compensator* of the default process establishes the cumulative rate of default of the structural model, which Giesecke and Goldberg (2004a, b) call the *pricing trend* (or simply *trend* in Giesecke 2005). This pricing trend characterizes the intensity of the default time, when it exits, providing the link between both approaches.

3 Compensators and pricing trends

In this section we shall derive some probabilistic results which are key in bridging the gap between structural and intensity models.

Consider a stopping time $\tau : \Omega \longrightarrow \mathcal{R}_+$ defined in our probability space, which represents the time of default. We say that a stopping time τ is predictable if there exists a sequence of stopping times which *announce* τ such that[5]

$$\begin{aligned} \tau_1 &\leq \tau_2 \leq ... < \tau, &(4)\\ \lim_{n\to\infty} \tau_n &= \tau. &(5) \end{aligned}$$

And τ is a totally inaccessible stopping time if there does not exist any predictable stopping time which can give us any information about τ, i.e.

$$\Pr\left[\tau = \tilde{\tau} < \infty\right] = 0 \qquad (6)$$

[5]Schönbucher (2003, Section 4.1) presents the following example. In a first passage model with complete information we can consider

$$\tau = \inf\left\{t \geq 0 \mid V_t \leq K\right\},$$

and a sequence of stopping times given by

$$\tau_n = \inf\left\{t \geq 0 \mid V_t \leq K + \frac{1}{n}\right\}.$$

for any predictable stopping time $\tilde{\tau}$.

The default indicator process N_t generated by τ is given by

$$N_t = \mathbf{1}_{\{\tau \le t\}}. \tag{7}$$

Definition 1 *A process C_t is called the $(\mathcal{F}_t)$-compensator of the process N_t if and only if the following conditions are satisfied:*

- *C_t is a $(\mathcal{F}_t)$-predictable increasing process, with $C_0 = 0$, and*
- *the process $N_t - C_t$, called the compensated process, follows a $(\mathcal{F}_t)$-martingale.*

As Bielecki and Rutkowski (2002, Section 4.6) note, the Doob-Meyer decomposition implies the existence of a unique $(\mathcal{F}_t)$-compensator C_t for the process N_t.

Giesecke (2005) and Giesecke and Goldberg (2004a, b) make use of a process Γ_t, referred to as *pricing trend*, associated with the $(\mathcal{F}_t)$-compensator C_t, such that

$$C_t = \Gamma_{\min\{t,\tau\}}. \tag{8}$$

The characteristic of this pricing trend is that the conditional default probability can be expressed as

$$P\left[\tau \le T \mid \mathcal{F}_t\right] = E\left[e^{\Gamma_t - \Gamma_T} \mid \mathcal{F}_t\right]. \tag{9}$$

Moreover, if we consider a defaultable security which pays X units at time T if default has not occurred before T and zero otherwise, we can express the price of the security at time $t \le T$ as

$$E\left[X e^{\Gamma_t - \Gamma_T - \int_t^T r_s ds} \mid \mathcal{F}_t\right], \tag{10}$$

where r_t is the $(\mathcal{F}_t)$-adapted interest rate process. The two previous expressions for the conditional default probability and the price of the defaultable securities are similar to those observed in an intensity model where, if λ_t is the intensity process, Γ_t would be the cumulative default process $\int_0^t \lambda_s ds$. The pricing trend Γ_t only admits an intensity representation when it is differentiable. In the cases where Γ_t is differentiable, there exists a process λ_t such that

$$\Gamma_t = \int_0^t \lambda_s ds \tag{11}$$

which represents the intensity of the counting process N_t, i.e. the intensity of arrival of the stopping time τ. Thus the pricing trend is the cumulatively default rate. But we should note again that Γ_t does not need to be differentiable. It is precisely here where the difference between reduced and intensity models lies. The reduced approach would consider default as an stopping time τ but without assuming the existence of an intensity process, i.e. reduced models do not assume that the pricing trend Γ_t is differentiable. Whereas an intensity model would be a reduced model in which the existence of an intensity λ_t is taken for granted. The inexistence of an intensity λ_t does not mean that we can not calculate default probabilities or price defaultable securities. To do that we just need the pricing trend.

The pricing trend Γ_t is characterized by a compensator process C_t, which is such that the difference between the default process N_t and the compensator follows a $(\mathcal{F}_t)$-martingale. If the filtration $(\mathcal{F}_t)$ represents the information the investors receive over time, different specifications of $(\mathcal{F}_t)$ will imply different compensator processes and then different pricing trends. Thus, pricing trends are determined by the specification of a stopping time τ and an information framework $(\mathcal{F}_t)$.

The last paragraph gives us the link between structural and reduced form models. A structural model is a way of specifying a default time τ based on the economic fundamentals of the firm, i.e.

$$\tau = \inf \{ t \geq 0 \mid V_t \leq K_t \} \tag{12}$$

where V_t and K_t are the firm's asset value and default threshold respectively. Equipped with a specification for the default time, each specification of the information $(\mathcal{F}_t)$ available to the investors, with respect to the asset value and default threshold processes, will yield a different pricing trend, and thus a different reduced-form model.

Therefore, informational assumptions are the key concept which provides us with a link between structural and reduced models.

The extreme case of a standard first passage model which implied predictable defaults does not admit an intensity of default. According to Giesecke (2005), a structural model in which investors have perfect information, the predictability of the default time implies that such models do not admit a conditional default rate, i.e. the default time does not have a conditional density. However, if we relax the complete information assumption, investors are uncertain about the true distance to default, and default comes as a surprise, which is a necessary condition for the model to admit a default rate. Thus, the unpredictability of default is a necessary, but not sufficient, condition for the pricing trend to admit an intensity. While any structural model with incomplete information admits a pricing trend, not all admit an intensity. The level of investors' information determines whether the model admits an intensity: it does not as soon as investors are certain about the firm's asset value. Giesecke (2005) provides sufficient conditions for the existence of a default intensity, and presents an extensive analysis of the consequences of several informational assumptions in the structural model.[6]

Duffie and Lando (2001), Çetin et al. (2004), Giesecke (2004, 2005) and Giesecke and Goldberg (2004a, b) deal with this approach for linking structural and reduced models, considering different specifications for the default time and for the information assumptions. We will comment some of them in the next section, but for a detailed study of this literature we refer the reader to the original papers.

4 Reconciliation models

Duffie and Lando (2001) consider a model in which the default time is fixed by the firm's managers in order to maximize the value of the equity, as in Leland and Toft (1996), considering a geometric Brownian motion for the asset process. However,

[6] See also Guo, Jarrow and Zeng (2005) for an analysis of the existence of a default intensity.

investors cannot observe the issuer's assets directly, and receive only periodic and imperfect accounting reports. Duffie and Lando derive the distribution of the firm's asset value conditional to investors' information and from it the intensity of default in terms of the conditional asset distribution and the default threshold. In this particular specification of the default time τ and the filtration $(\mathcal{F}_t)$, an intensity for the default time can be derived. Duffie and Lando (2001) represent one of the first attempts to link structural and reduced form models, and although they do not use explicitly the concepts of compensator and pricing trend, their approach is an equivalent one.

Giesecke (2005) deals first with the case of a structural model in which investors have complete information about both the level of the firm's asset value and the default threshold. Considering a continuous process for the asset value, we are in the case of a standard first passage model which implies predictable defaults. After that, he deals with the case of complete information about the asset value but incomplete information about the default threshold. Although constant, the default threshold is not known by the investors, who are forced to work under a distribution function for the default threshold. The impossibility of observing the default threshold makes the default time an unpredictable event. In this case, investors can calculate the pricing trend in terms of the distribution function for the threshold and the observable historical asset value. However, in this specification of the information setting, the pricing trend does not admit a intensity of default.

Next, Giesecke studies the cases of incomplete information for the asset value (instead of the default threshold) and for both the asset value and the default threshold. In contrast with the previous case in which investors have incomplete information about the default threshold but complete information about the asset value process, with imperfect asset information the pricing trend, calculated in terms of the threshold distribution and the distribution for the minimum historical asset level, admits an intensity representation.

Giesecke and Goldberg (2004a) consider the case in which the default barrier is random and unobserved, which they argue (p. 12) "is consistent with recent experiences at Enron, WorldCom, and Tyco, which surely show that investors cannot observe the barrier. In each of these cases, the true level of liabilities was not disclosed to the public." Not knowing the default threshold, investors use a priory distribution for its value.

Giesecke (2004) takes the incomplete information assumption in structural model one step further: the modelling of default correlation. He provides a structural model in which the firms' default probabilities are linked via a joint distribution for their default thresholds. Investors do not have perfect information about neither such thresholds nor about their joint distribution. However, they form a prior distribution which is updated at any time one of such thresholds is revealed, which only happens when one of the firms defaults. In Giesecke (2004) investors have incomplete information about the firms' default thresholds but complete information about their asset processes. Giesecke and Goldberg (2004b) extend that framework to one in which investors do not have information neither about the firms' asset values nor about their default thresholds. In this case, default correlation is introduced through correlated asset processes and, again, investors receive information about the firms' asset and default barrier only when they default. Such information is used to update their priors about the distribution of the remaining firms' asset values.[7]

Çetin et al. (2004) propose a similar approach for linking structural and intensity models to the one by Duffie and Lando, assuming that investors receive only a reduced version of the information that firm's managers have available. They claim that the default time is a predictable event for firm's managers, since they have enough information about the firm's fundamentals. But public investors do not have access to that information. Instead they observe a reduced version of this information. In their model, the firm's Cash Flow (CF) is the variable which triggers default, after reaching some minimum levels during a given period of time. Firm's managers can see the CF levels, but investors only receive information about the sign of the CF,

[7]See also Guo, Jarrow and Zeng (2005) and Giesecke and Goldberg (2005) for extensions to default correlation and contagion.

making the default time an unpredictable event for them. In this setting, they derive the default intensity as seen by the market. As Duffie and Lando, they do not make use of neither the compensator nor the pricing trend process.

Schönbucher (1996) presents a slightly different way of bridging the gap between structural and reduced models, based on the introduction of a jump in the firm's asset value process. However, Duffie and Lando (2001, p. 635-6) argue that this approach, although solves the problem of zero short term spreads in structural models, does not theoretically bridge the gap between structural and reduced form models because it is not (generally) consistent with a default intensity.

Guo, Jarrow and Zeng (2005) consider that the ways in which the previous papers introduce incomplete information about the variables generating default are illustrative but too simple to be applied in practice. Their paper represents an extension and generalization of the previous model, in an attempt to formalize the theory linking reduced and structural models.

In light of the differences in terms of information assumptions between reduced and structural model, Jarrow and Protter (2004, p. 2) argue in favor of the usage of reduced form models:

> "An immediate consequence of this observation is that the current debate in the credit risk literature about these two model types is misdirected. Rather than debating which model type is best in terms of forecasting performance, the debate should be focused on whether the model should be based on the information set observed by the market or not. For pricing and hedging credit risk, we believe that the information set observed by the market is the relevant one. This is the information set used by the market, in equilibrium, to determine prices. Given that belief, a reduced form model should be employed.
>
> [...] Surprisingly, at this stage in the credit risk literature, there appears to be no disagreement that the asset value process is unobservable by the

market (see especially: Duan, 1994; Ericsson and Reneby, 2002, 2003 in this regard). Although not well understood in terms of its implication, this consensus supports the usage of reduced form models."

References

[1] Bielecki, T. R., and Rutkowski, M., 2002, "Credit Risk: Modeling, Valuation and Hedging," *Springer Finance.*

[2] Brody, D., Hughston, L., and Macrina, A., 2005, "Beyond Hazard Rates: a New Framework for Credit-Risk Modelling," available at `www.mth.kcl.ac.uk/~avraam/bhm.pdf`.

[3] Çetin, U., Jarrow, R., Protter, P., and Yildirim, Y., 2004, "Modelling Credit Risk with Partial Information," The Annals of Applied Probability, 14, 1167-1178.

[4] Duan, J. C., 1994, "Maximum Likelihood Estimation using Price Data of the Derivative Contract," *Mathematical Finance* 5, 155-167.

[5] Duffie, D., and Lando, D., 2001, "Term Structure of Credit Spreads with Incomplete Accounting Information," *Econometrica* 69, 633-664.

[6] Duffie, D., and Singleton, K. J., 2003, "Credit Risk: Pricing, Measurement and Management," *Princeton Series in Finance.*

[7] Elizalde, A., 2005a, "Credit Risk Models I: Default Correlation in Intensity Models," available at `www.abelelizalde.com`.

[8] Elizalde, A., 2005b, "Credit Risk Models II: Structural Models," available at `www.abelelizalde.com`.

[9] Ericsson, J., and Reneby, J., 2002, "Estimating Structural Bond Pricing Models," Working Paper, Stockholm School of Economics.

[10] Ericsson, J., and Reneby, J., 2003, "The Valuation of Corporate Liabilities: Theory and Tests," Working Paper, SSE/EFI No. 445.

[11] Giesecke, K., 2004, "Correlated default with incomplete information," *Journal of Banking and Finance* 28, 1521-1545.

[12] Giesecke, K., 2005, "Default and Information," Working Paper, Cornell University.

[13] Giesecke, K., and Goldberg, L., 2004a, "Forecasting default in the face of uncertainty," *Journal of Derivatives* 12, 11-25.

[14] Giesecke, K., and Goldberg, L., 2004b, "Sequential defaults and incomplete information," *Journal of Risk* 7, 1-26.

[15] Giesecke, K., and Goldberg, L., 2005, "A Top Down Approach to Multi-Name Credit."

[16] Guo, X., Jarrow, R., and Zeng, Y., 2005, "Information Reduction in Credit Risk Models."

[17] Jarrow, R. A., and Protter, P., 2004, "Structural versus reduced form models: a new information based perspective," *Journal of Investment Management* 2, 1-10.

[18] Leland, H. E., and Toft, K. B., 1996, "Optimal Capital Structure, Endogenous Bankruptcy and the Term Structure of Credit Spreads," *Journal of Finance* 50, 789-819.

[19] Protter, P., 1990, "Stochastic Differentiation and Differential Equations. A New Approach," Springer-Verlag.

[20] Schönbucher, P. J., 1996, "Valuation of Securities Subject to Credit Risk," Working Paper, University of Bonn, Department of Statistics..

[21] Schönbucher, P. J., 2003, "Credit Derivatives Pricing Models," *Wiley Finance*.